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Diffraction of internal waves by a submerged circular cylinder at forward speed in a two-layer fluid

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Abstract. Diffraction of internal waves by a submerged body in a uniform current of a two-layer fluid is considered. The layers are infinitely deep, and the flows are two-dimensional. The linearized potential theory is used for the inviscid and incompressible fluid. The solution for the circular cylinder, which is either below or above the interface, is given in the form of rapidly converging series. This is achieved through the use of certain recursive relations. Numerical results are provided for the exciting forces, wave resistance and lift which may be useful in testing numerical methods used for the study of internal wave diffraction by a submerged body of arbitrary form.

Keywords: two-layer fluid, diffraction problem, flow, circular cylinder, hydrodynamic loads

1. Introduction

Sea-wave propagation in the presence of different underwater obstacles and diffraction effect are matters of great importance for water engineering and underwater navigation. The problem of wave scattering by a submerged body has been investigated in detail for regular linear surface waves; however, significant scattering of internal waves may occur when underwater objects are placed near the region of high-gradient density. The simplest example of stratified fluid is the two-layer fluid. A particular case is a homogeneous fluid with a free surface, as the air density is assumed negligible compared with the water density.

The linear problem of surface-wave diffraction was studied both for a restricted and uniformly moving submerged body. This problem belongs to the linear theory of seakeeping. In the two-dimensional case the linear theory of seakeeping has been studied in detail for a circular cylinder submerged in deep water.

Grue and Palm [1], and then Grue [2] pioneered the solution of the radiation and diffraction problems of a submerged circular cylinder in a uniform current. They used the source-distribution method. Almost the same problem was considered by Kashiwagi *et al.* [3], but they used the integral-equation method for the velocity potential on the body surface and the first-order approximation for the steady potential: the infinite-fluid solution valid for a ‘deeply’ submerged body. The multipole expansion method was recently adopted by Wu [4]. In his paper the numerical results of the steady, radiation and diffraction loads are tabulated. In principle, in all above-mentioned papers the inversion of the infinite matrix was required for obtaining the final results.

In contrast to these methods, Mehlum [5] obtained an explicit solution in the form of rapidly converging series for wave diffraction by a submerged cylinder without forward speed. The practical computation of the velocity potential is reduced almost to hand calculations. A similar method was proposed by Sretensky [6] for the steady problem on a uniformly moving submerged circular cylinder. Unfortunately, his paper appeared in a relatively inaccessible periodical and has remained practically unknown.

The results of [5] were extended to the two-layer infinite fluid for the diffraction problem without forward speed by Khabakhpasheva [7], and, later on, for the steady problem by Khabakhpasheva [8]. In both cases a circular cylinder is in the lower layer.

The aim of the present paper is to derive an explicit solution for internal wave diffraction by a circular cylinder located under or above the interface in a uniform current. The paper is organized as follows. Section 2 introduces the governing equations. In Section 3 the diffraction potentials and exciting forces for the cylinder submerged in the lower layer are presented. In Section 4 similar results are obtained for a cylinder in the upper layer. The tables of exciting forces for the homogeneous and two-layer fluid are given. For determining the exciting forces in a diffraction problem with forward speed one needs a solution of the steady problem too. Appendix A gives a brief solution of the steady problem for a cylinder both in the lower layer and in the upper layer. The tables of the wave resistance and lift are presented. In Appendix B the details of the special integrals used here are given.

2. Governing equations

Let a Cartesian coordinate system be taken with the x_0-axis directed along an equilibrium position of the interface in the direction of forward speed U, perpendicular to a cylinder axis, and the y_0-axis pointing vertically upwards. The coordinate system moves with the body at the same speed. In the undisturbed state, the upper fluid layer with the density ρ_1 occupies the domain $|x_0| < \infty, y_0 > 0$, and the lower one with the density $\rho_2 = \rho_1(1 + \varepsilon)$ $(\varepsilon > 0)$, the domain $|x_0| < \infty, y_0 < 0$. We assume the fluid to be inviscid and incompressible, and the disturbance of the interface to be small. The flows in each layer are potential. For a time-periodic incoming wave at a frequency ω_0 the total velocity potential can be written as

$$\Phi^{(s)}(x_0, y_0, t) = -Ux_0 + U\bar{\Phi}^{(s)}(x_0, y_0) + \operatorname{Re}\{\eta_0[\Phi_0^{(s)}(x_0, y_0) + \Phi_1^{(s)}(x_0, y_0)]\, \mathrm{e}^{i\omega t}\},$$

where $\bar{\Phi}^{(s)}$ is the steady potential due to the unit forward speed; $\Phi_0^{(s)}$ and $\Phi_1^{(s)}$ are the potentials of the incident and diffracted waves, respectively; and η_0 is the amplitude of the incoming wave. The superscript s is equal to 1 for the upper layer and 2 for the lower one. The encounter frequency ω is obtained from

$$\omega = |\omega_0 \pm k_0 U|, \qquad k_0 = \omega_0^2/\bar{g}, \qquad \bar{g} = \varepsilon g/(2 + \varepsilon), \tag{2.1}$$

where g is the acceleration due to gravity; the signs '+' and '−' correspond to waves travelling from the right and from the left, respectively.

The incident potentials are

$$\Phi_0^{(s)} = i\sqrt{\bar{g}/k_0}\,\phi_0^{(s)} \exp(\pm i k_0 x_0), \qquad \phi_0^{(1)}(y_0) = -\mathrm{e}^{-k_0 y_0}, \qquad \phi_0^{(2)}(y_0) = \mathrm{e}^{k_0 y_0}. \tag{2.2}$$

Based on the assumptions of linear potential flow theory, the governing equations for the steady potential are

$$\nabla^2\bar{\Phi}^{(1)} \equiv \partial^2\bar{\Phi}^{(1)}/\partial x_0^2 + \partial^2\bar{\Phi}^{(1)}/\partial y_0^2 = 0 \quad (y_0 > 0), \qquad \nabla^2\bar{\Phi}^{(2)} = 0 \quad (y_0 < 0) \tag{2.3}$$

with boundary conditions

$$\nabla\bar{\Phi}^{(1)} \to 0 \quad (y_0 \to \infty), \qquad \nabla\bar{\Phi}^{(2)} \to 0 \quad (y_0 \to -\infty), \tag{2.4}$$

$$\partial\bar{\Phi}^{(q)}/\partial n = n_1 \quad ((x_0, y_0) \in S), \tag{2.5}$$

where S is the surface of the cylinder, $\partial/\partial n$ denotes the normal derivative and n_1 is the x_0-component of the unit normal vector $\mathbf{n} = (n_1, n_2)$ pointing into the body. Here $q = 1 (q = 2)$ if the cylinder is in the upper (lower) layer. The linearized dynamic and kinematic boundary conditions on the interface are

$$(1+\varepsilon)\frac{\partial^2\bar{\Phi}^{(2)}}{\partial x_0^2} - \frac{\partial^2\bar{\Phi}^{(1)}}{\partial x_0^2} + \frac{\varepsilon g}{U^2}\frac{\partial\bar{\Phi}^{(1)}}{\partial y_0} = 0, \qquad \frac{\partial\bar{\Phi}^{(1)}}{\partial y_0} = \frac{\partial\bar{\Phi}^{(2)}}{\partial y_0}, \quad (y_0 = 0), \tag{2.6}$$

respectively. We also adopt the radiation condition which assumes that there is no wave due to steady potentials far in front of the cylinder at $x \to \infty$.

The diffraction potentials satisfy equations similar to (2.3)

$$\nabla^2\Phi_1^{(1)} = 0 \quad (y_0 > 0), \qquad \nabla^2\Phi_1^{(2)} = 0 \quad (y_0 < 0)$$

with boundary conditions

$$\nabla\Phi_1^{(1)} \to 0 \quad (y_0 \to \infty), \qquad \nabla\Phi_1^{(2)} \to 0 \quad (y_0 \to -\infty),$$

$$\partial\Phi_1^{(q)}/\partial n = -\partial\Phi_0^{(q)}/\partial n \quad ((x_0, y_0) \in S), \tag{2.7}$$

$$\left(U\frac{\partial}{\partial x_0} - i\omega\right)^2 [(1+\varepsilon)\Phi_1^{(2)} - \Phi_1^{(1)}] + \varepsilon g\frac{\partial\Phi_1^{(1)}}{\partial y_0} = 0,$$

$$\frac{\partial\Phi_1^{(1)}}{\partial y_0} = \frac{\partial\Phi_1^{(2)}}{\partial y_0} \quad (y_0 = 0). \tag{2.8}$$

The radiation condition for $\Phi_1^{(s)}$ states that a wave travelling in the direction of the forward speed, and with its group velocity larger than the forward speed, propagates to $x \to \infty$, and otherwise the waves propagate to $x \to -\infty$.

It is convenient to introduce new unknown functions $\Upsilon^{(s)}$ and $\Psi^{(s)}$, where

$$\Upsilon^{(s)}(x_0, y_0) = \bar{\Phi}^{(s)} - x_0, \qquad \Psi^{(s)}(x_0, y_0) = \phi_0^{(s)}\exp(\pm ik_0x_0) - i\sqrt{k_0/g}\,\Phi_1^{(s)}. \tag{2.9}$$

According to the boundary conditions (2.5) and (2.7) both these functions have zero normal derivatives on the surface of the cylinder. The dynamic boundary conditions on the interface for $\Upsilon^{(s)}$ and $\Psi^{(s)}$ have the same form as (2.6) and (2.8) for $\bar{\Phi}^{(s)}$ and $\Phi_1^{(s)}$, respectively, because the potentials of the incident waves $\phi_0^{(s)}$ in (2.2) satisfy (2.8).

After the steady and diffraction potentials have been obtained, the steady and exciting forces can be computed from (Newman [9])

$$F_{sj} = -\rho_q U^2 \int_S \left(\frac{\partial\bar{\Phi}^{(q)}}{\partial x_0} - \tfrac{1}{2}|\nabla\bar{\Phi}^{(q)}|^2\right) n_j \,\mathrm{d}s, \tag{2.10}$$

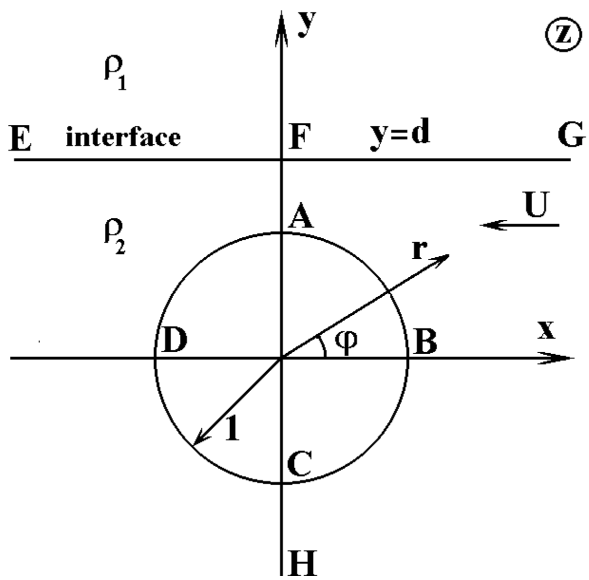

Figure 1. Submerged circular cylinder under an interface.

$$F_{ej} = \rho_q \eta_0 \int_S [i\omega(\Phi_0^{(q)} + \Phi_1^{(q)}) + U\nabla(\bar{\Phi}^{(q)} - x_0)\nabla(\Phi_0^{(q)} + \Phi_1^{(q)})] n_j \, \mathrm{d}s$$

$$= \rho_q \eta_0 \sqrt{\frac{\bar{g}}{k_0}} \int_S (iU\nabla\Upsilon^{(q)}\nabla\Psi^{(q)} - \omega\Psi^{(q)}) n_j \, \mathrm{d}s, \quad (j = 1, 2), \tag{2.11}$$

where F_{s1} is the wave resistance, F_{s2} is the lift, F_{e1} and F_{e2} indicate the horizontal and vertical exciting forces.

The centre of the cylinder is located at $y_0 = -h(y_0 = h)$ for the body submerged in the lower (upper) layer. With a being the radius of the cylinder, we have $h > a$. We can now scale the coordinates so that the dimensionless cylinder radius is equal to unity:

$$x = x_0/a, \quad y_1 = y_0/a, \quad d = h/a > 1, \quad k = k_0 a.$$

3. A circular cylinder submerged in the lower fluid

Let us transfer the origin of the coordinates into the cylinder center obeying the translation $y = y_1 + d$. The geometry of flow is shown in Figure 1. We introduce the new coordinates $w = u + iv = \rho\, \mathrm{e}^{i\theta}$ by means of a conformal, bilinear mapping

$$w = \frac{i - Rz}{R + iz}, \tag{3.1}$$

where $z = r\, \mathrm{e}^{i\varphi} = x + iy, \quad R = d - \gamma, \quad \gamma = \sqrt{d^2 - 1}$.

The fluid is now contained in the circular region, shown in Figure 2. The cylinder surface is the circle $|w| = 1$, while the interface is the circle $|w| = R$. The upper layer is contained in the circular region $|w| < R$, and the lower layer is contained in the annular region $R < |w| < 1$.

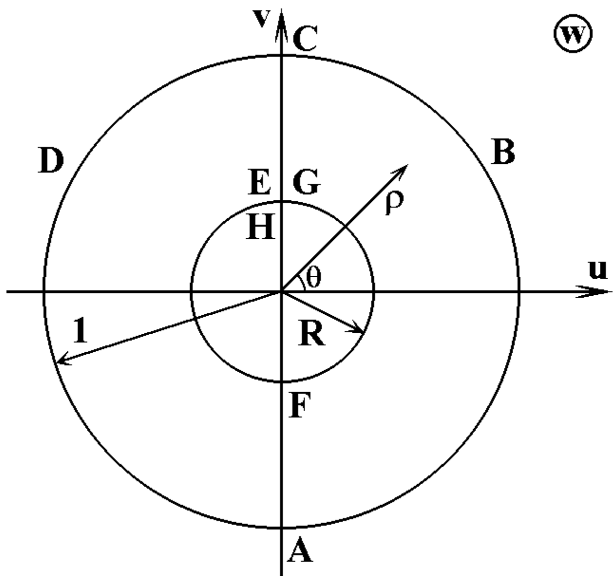

Figure 2. The image of the flow region shown in Figure 1.

Figure 1 and Figure 2 show the correspondence of certain points of the planes z and w. The specific feature of the mapping (3.1) is that all points at infinity $z \to \infty$ are mapped onto one point $w \to iR$ (*cf.* points E, G, H, K in Figure 1 and Figure 2).

The steady problem solution for this case is presented in detail by Khabakhpasheva [8]. In Appendix A the basic results are briefly described with corrections of some inaccuracies.

3.1. The diffraction potentials

Let us express the potentials $\Psi^{(s)}$ representing the sum of the incident and diffracted potentials as the series based on the system of fundamental solutions of the Laplace equation for annular regions, taking into account that the normal derivative $\Psi^{(2)}$ on the cylinder surface is equal to zero and $\Psi^{(1)}$ is the analytical function in the circle $|w| < R$

$$\Psi^{(1)} = \sum_{n=-\infty}^{\infty} B_n \rho^{|n|} \, \mathrm{e}^{in\theta}, \qquad \Psi^{(2)} = \sum_{n=-\infty}^{\infty} C_n (\rho^n + \rho^{-n}) \, \mathrm{e}^{in\theta}. \tag{3.2}$$

Applying the kinematic boundary condition in (2.8), we can express the coefficients B_n through C_n at $n \neq 0$. The potentials in (3.2) can be represented as

$$\Psi^{(s)} = \Pi_1^{(s)} + \Pi_2^{(s)} \quad (s = 1, 2), \tag{3.3}$$

where

$$\Pi_1^{(1)}(\rho, \theta) \;=\; B_0/2 + \sum_{n=1}^{\infty} C_{-n} R_n^- R^{-n} \rho^n \, \mathrm{e}^{-in\theta},$$

$$\Pi_2^{(1)}(\rho, \theta) \;=\; B_0/2 + \sum_{n=1}^{\infty} C_n R_n^- R^{-n} \rho^n \, \mathrm{e}^{in\theta}, \tag{3.4}$$

$$\Pi_1^{(2)}(\rho,\theta) = C_0 + \sum_{n=1}^{\infty} C_{-n}(\rho^n + \rho^{-n})\,\mathrm{e}^{-in\theta},$$

$$\Pi_2^{(2)}(\rho,\theta) = C_0 + \sum_{n=1}^{\infty} C_n(\rho^n + \rho^{-n})\,\mathrm{e}^{in\theta}, \tag{3.5}$$

$$R_n^- = R^n - R^{-n}.$$

The dynamic boundary condition on the interface (2.8) has the form

$$\frac{\tau^2}{\nu}\left[\frac{\partial^2\Psi_\varepsilon}{\partial\theta^2}\left(\frac{\partial\theta}{\partial x}\right)^2 + \frac{\partial\Psi_\varepsilon}{\partial\theta}\frac{\partial^2\theta}{\partial x^2}\right] - 2i\tau\frac{\partial\Psi_\varepsilon}{\partial\theta}\frac{\partial\theta}{\partial x} - \nu\Psi_\varepsilon$$

$$+\varepsilon\frac{\partial\Psi^{(2)}}{\partial\rho}\frac{\partial\rho}{\partial y} = 0 \quad (\rho = R), \tag{3.6}$$

where

$$\Psi_\varepsilon = (1+\varepsilon)\Psi^{(2)} - \Psi^{(1)},$$

$$\frac{\partial\rho}{\partial y} = -R\frac{\partial\theta}{\partial x} = \frac{R}{\gamma}(\sin\theta - 1), \qquad \frac{\partial^2\theta}{\partial x^2} = \frac{1}{2\gamma^2}(3 - \cos 2\theta - 4\sin\theta), \tag{3.7}$$

$$\tau = U\omega/g, \quad \nu = \omega^2 a/g.$$

Substituting (3.3) in (3.6) and matching the coefficients in front of all terms $\mathrm{e}^{in\theta}$ we get the system of recursive relations to determine the coefficients in the expansion (3.2).

It is convenient to consider the solution for $\Pi_1^{(s)}$ (*case 1*) and $\Pi_2^{(s)}$ (*case 2*) further in its own right.

(a) *Case 1*.

The recursive relations for the coefficients C_{-n} are

$$C_{2-n}\frac{\tau^2}{2\nu\gamma}(n-2)(n-1)P_{n-2}$$

$$+C_{1-n}i(n-1)\left\{\left[2 + \frac{\tau}{\nu\gamma}(2n-1)\right]\tau P_{n-1} + \varepsilon R_{n-1}^-\right\}$$

$$-C_{-n}\left[\left(4n\tau + 2\nu\gamma + \frac{3n^2\tau^2}{\nu\gamma}\right)P_n + 2\varepsilon n R_n^-\right]$$

$$-C_{-n-1}i(n+1)\left\{\left[2 + \frac{\tau}{\nu\gamma}(2n+1)\right]\tau P_{n+1} + \varepsilon R_{n+1}^-\right\}$$

$$+C_{-n-2}\frac{\tau^2}{2\nu\gamma}(n+2)(n+1)P_{n+2} = 0 \quad (n \geqslant 1), \tag{3.8}$$

where

$$P_n = \varepsilon R^n + (2+\varepsilon) R^{-n}.$$

Once C_{-1} and C_{-2} are given, (3.8) allows an explicit evaluation of all coefficients $C_{-n}(n > 2)$. Up to now, we do not consider the terms which are independent of θ. This case in point is at the end of this section.

The series (3.4), (3.5) will, however, show bad convergence for the points x, y far away from the cylinder, and these series are not suitable for analysis of the asymptotic behaviour of the potentials (see [5] for more details). Therefore, we need to work out an equivalent solution which gives direct information about the asymptotic behaviour of the potentials as $|x| \to \infty$. Using the boundary conditions at $|x| \to \infty$, we will define C_{-1} and C_{-2}. To do this we introduce new functions

$$G_1(\xi) = \sum_{n=1}^{\infty} C_{-n} R^n \xi^n, \qquad G_2(\xi) = \sum_{n=1}^{\infty} C_{-n} R^{-n} \xi^n \tag{3.9}$$

and their sum with weights

$$W_1(\xi) = \varepsilon G_1(\xi) + (2+\varepsilon) G_2(\xi). \tag{3.10}$$

Multiplying the recursive relations (3.8) by ξ^n and summing them with respect to n from 1 to ∞, we obtain the following differential equation

$$\begin{aligned} &\frac{\bar{\tau}^2}{2\bar{\nu}\gamma}(1-i\xi)^4 W_1'' - i(1-i\xi)^2 \left[\frac{\bar{\tau}^2}{\bar{\nu}\gamma}(1-i\xi) + 2\bar{\tau} - 1\right] W_1' - 2\bar{\nu}\gamma W_1 \\ &\quad = K_1 + 2i(1+\varepsilon)(1-i\xi)^2 G_1', \end{aligned}$$

where

$$\begin{aligned} K_1 \;&=\; \frac{\bar{\tau}^2}{\bar{\nu}\gamma} C_{-2} P_2 - i C_{-1}\left[\left(\frac{\bar{\tau}}{\bar{\nu}\gamma} + 2\right)\bar{\tau} P_1 + (2+\varepsilon) R_1^-\right], \\ &(\bar{\tau}, \bar{\nu}) = \frac{2+\varepsilon}{\varepsilon}(\tau, \nu). \end{aligned} \tag{3.11}$$

A prime denotes differentiation with respect to ξ. The general solution of this equation is given by

$$\begin{aligned} W_1 \;=\; &\frac{\bar{\nu}(1+\varepsilon)}{\bar{\tau}^2(k_1-k_2)} \sum_{n=1}^{\infty} C_{-n} R^n [I_n(\gamma k_1, \xi) - I_n(\gamma k_2, \xi)] - c_1 \exp\left(-\frac{2\gamma k_2}{1-i\xi}\right) \\ &+ c_2 \exp\left(-\frac{2\gamma k_1}{1-i\xi}\right) - \frac{K_1}{2\gamma\bar{\nu}}, \end{aligned} \tag{3.12}$$

where

$$k_{1,2} = \frac{\bar{\nu}}{2\bar{\tau}^2}(1 - 2\bar{\tau} \pm \sqrt{1-4\bar{\tau}}), \tag{3.13}$$

$$I_n(\beta,\xi) = 4i\beta \exp\left(-\frac{2\beta}{1-i\xi}\right)\int_0^{\xi}\frac{t^n}{(1-it)^2}\exp\left(\frac{2\beta}{1-it}\right)\,\mathrm{d}t \tag{3.14}$$

and c_1 and c_2 are constants of integration. Real values k_1 and k_2 are possible at $\bar{\tau} \leqslant \frac{1}{4}$ only. The constants c_1, c_2 are linearly dependent on C_{-1} and C_{-2}. Indeed, Equation (3.10) give $W_1(0) = 0$, $W_1'(0) = C_{-1}P_1$, and according to (3.12)

$$W_1(0) = c_2\,\mathrm{e}^{-2\gamma k_1} - c_1\,\mathrm{e}^{-2\gamma k_2} - K_1/(2\gamma\bar{\nu}),$$
$$W_1'(0) = 2i\gamma(k_2c_1\,\mathrm{e}^{-2\gamma k_2} - k_1c_2\,\mathrm{e}^{-2\gamma k_1}).$$

Consequently, from the solution of the resulting linear second-order system we get c_1 and c_2 for further application in the transformed form

$$(c_1, c_2) = (2+\varepsilon)(\bar{c}_1\,\mathrm{e}^{2\gamma k_2}, \bar{c}_2\,\mathrm{e}^{2\gamma k_1}),$$

where

$$\bar{c}_1 = a_1C_{-1} + b_1C_{-2}, \qquad \bar{c}_2 = a_2C_{-1} + b_2C_{-2},$$
$$a_q = \frac{1}{(2+\varepsilon)(k_1-k_2)}\left(\frac{iP_1}{2\gamma} - \frac{v_1k_q}{\bar{\nu}}\right), \qquad b_q = -\frac{k_qv_2}{\bar{\nu}(2+\varepsilon)(k_1-k_2)} \quad (q=1,2),$$
$$v_1 = -\frac{i}{2\gamma}\left[\left(2+\frac{\bar{\tau}}{\bar{\nu}\gamma}\right)\bar{\tau}P_1 + (2+\varepsilon)R_1^-\right], \quad v_2 = \frac{\bar{\tau}^2}{2\bar{\nu}\gamma^2}P_2. \tag{3.15}$$

(b) *Case 2.*
The solution for $\Pi_2^{(s)}$ is derived in a similar way. The recursive relations for C_n in (3.4), (3.5) are

$$\begin{aligned}
&C_{n-2}\frac{\tau^2}{2\nu\gamma}(n-2)(n-1)P_{n-2}\\
&\quad + C_{n-1}i(n-1)\left\{\left[2-\frac{\tau}{\nu\gamma}(2n-1)\right]\tau P_{n-1} - \varepsilon R_{n-1}^-\right\}\\
&\quad + C_n\left[\left(4n\tau - 2\nu\gamma - \frac{3n^2\tau^2}{\nu\gamma}\right)P_n - 2\varepsilon nR_n^-\right]\\
&\quad + C_{n+1}i(n+1)\left\{\left[\frac{\tau(2n+1)}{\nu\gamma} - 2\right]\tau P_{n+1} + \varepsilon R_{n+1}^-\right\}\\
&\quad + C_{n+2}\frac{\tau^2}{2\nu\gamma}(n+2)(n+1)P_{n+2} = 0 \quad (n \geqslant 1),
\end{aligned} \tag{3.16}$$

Once C_1 and C_2 have been defined, the series for $\Pi_2^{(s)}$ are determined except for the coefficients B_0 and C_0, which are evaluated below. The values of C_1 and C_2 can be found with the help of the boundary conditions at $|x| \to \infty$. We introduce new functions

$$F_1(\xi) = \sum_{n=1}^{\infty} C_nR^n\xi^n, \qquad F_2(\xi) = \sum_{n=1}^{\infty} C_nR^{-n}\xi^n, \tag{3.17}$$

$$W_2(\xi) = \varepsilon F_1(\xi) + (2+\varepsilon) F_2(\xi) \tag{3.18}$$

and obtain a differential equation

$$\begin{aligned}\frac{\bar\tau^2}{2\bar\nu\gamma}(1+i\xi)^4 W_2'' + i(1+i\xi)^2\left[\frac{\bar\tau^2}{\bar\nu\gamma}(1+i\xi) - 2\bar\tau - 1\right] W_2' - 2\bar\nu\gamma W_2 \\ = K_2 - 2i(1+\varepsilon)(1+i\xi)^2 F_1',\end{aligned} \tag{3.19}$$

where

$$K_2 = \frac{\bar\tau^2}{\bar\nu\gamma} C_2 P_2 + iC_1\left[\left(\frac{\bar\tau}{\bar\nu\gamma} - 2\right)\bar\tau P_1 + (2+\varepsilon) R_1^-\right].$$

The general solution of Equation (3.19) is

$$\begin{aligned}W_2 \;=\; & \frac{\bar\nu(1+\varepsilon)}{\bar\tau^2(k_3-k_4)} \sum_{n=1}^{\infty} C_n R^n [J_n(\gamma k_4, \xi) - J_n(\gamma k_3, \xi)] + c_3 \exp\left(-\frac{2\gamma k_4}{1+i\xi}\right) \\ & + c_4 \exp\left(-\frac{2\gamma k_3}{1+i\xi}\right) - \frac{K_2}{2\gamma\bar\nu},\end{aligned} \tag{3.20}$$

where

$$k_{3,4} = \frac{\bar\nu}{2\bar\tau^2}(1 + 2\bar\tau \pm \sqrt{1+4\bar\tau}), \tag{3.21}$$

$$J_n(\beta,\xi) = 4i\beta \exp\left(-\frac{2\beta}{1+i\xi}\right)\int_0^{\xi} \frac{t^n}{(1+it)^2}\exp\left(\frac{2\beta}{1+it}\right)\mathrm{d}t \tag{3.22}$$

and c_3 and c_4 are constants of integration, which are linearly dependent on C_1 and C_2 similar to *case 1*. According to (3.18) we have $W_2(0) = 0$, $W_2'(0) = C_1 P_1$, and according to (3.20)

$$W_2(0) = c_3\, \mathrm{e}^{-2\gamma k_4} + c_4\, \mathrm{e}^{-2\gamma k_3} - K_2/(2\gamma\bar\nu),$$

$$W_2'(0) = 2i\gamma(k_4 c_3\, \mathrm{e}^{-2\gamma k_4} + k_3 c_4\, \mathrm{e}^{-2\gamma k_3}).$$

Consequently, from the solution of the resulting linear second-order system, we get c_3 and c_4

$$(c_3, c_4) = (2+\varepsilon)(\bar c_3\, \mathrm{e}^{2\gamma k_4}, \bar c_4\, \mathrm{e}^{2\gamma k_3}),$$

where

$$\bar c_3 = a_3 C_1 + b_3 C_2, \qquad \bar c_4 = -(a_4 C_1 + b_4 C_2),$$

$$a_q = \frac{1}{(2+\varepsilon)(k_3-k_4)}\left(\frac{iP_1}{2\gamma} + \frac{\upsilon_3 k_q}{\bar\nu}\right), \qquad b_q = \frac{k_q \upsilon_2}{\bar\nu(2+\varepsilon)(k_3-k_4)} \quad (q = 3, 4),$$

$$\upsilon_3 = \frac{i}{2\gamma}\left[\left(\frac{\bar\tau}{\bar\nu\gamma} - 2\right)\bar\tau P_1 + (2+\varepsilon) R_1^-\right]$$

and where v_2 is defined in (3.15). Furthermore, taking $\xi = R\rho^{-1}\,\mathrm{e}^{-i\theta}$ in (3.12) and $\xi = R\rho^{-1}\,\mathrm{e}^{i\theta}$ in (3.20), we get

$$W_1 = \sum_{n=1}^{\infty} C_{-n}[\varepsilon(R^{2n}+1)+2]\rho^{-n}\,\mathrm{e}^{-in\theta},$$

$$W_2 = \sum_{n=1}^{\infty} C_{n}[\varepsilon(R^{2n}+1)+2]\rho^{-n}\,\mathrm{e}^{in\theta}. \tag{3.23}$$

We shall consider the potential of the lower layer $\Psi^{(2)}$ at the interface. According to (3.5) at $\rho = R$ we have

$$\Pi_1^{(2)}(R,\theta) = C_0 + \sum_{n=1}^{\infty} C_{-n}R_n^{+}\,\mathrm{e}^{-in\theta}, \qquad \Pi_2^{(2)}(R,\theta) = C_0 + \sum_{n=1}^{\infty} C_{n}R_n^{+}\,\mathrm{e}^{in\theta}, \tag{3.24}$$

where $R_n^{+} = R^n + R^{-n}$. Using (3.23) at $\rho = R$, we obtain

$$\Pi_1^{(2)}(R,\theta) = C_0 + (W_1 + 2G_1)/(2+\varepsilon), \qquad \Pi_2^{(2)}(R,\theta) = C_0 + (W_2 + 2F_1)/(2+\varepsilon).$$

In order to determine the behaviour $\Pi_{1,2}^{(2)}$ at $\theta \to \pi/2$ ($\xi \to i$), which is equivalent to $|x| \to \infty$, it is necessary to investigate the properties of the integral functions $J_n(\beta,\xi)$ and $I_n(\beta,\xi)$. From (3.14) and (3.22) it is easy to get $I_n(\beta,\xi) = -J_n^{*}(\beta,\xi)$, where the symbol $*$ denotes the complex conjugate. Computation of the integrals J_n is fully considered in [5], and the basic results are briefly described in Appendix B.

Using these results, we can present the potentials $\Pi_{1,2}^{(2)}$ at $|x| \to \infty$ as follows

$$\Pi_1^{(2)} = (\bar{c}_2 - \sigma_1 A_1)\,\mathrm{e}^{k_1(y-ix-R)} + (\sigma_1 A_2 - \bar{c}_1)\,\mathrm{e}^{k_2(y-ix-R)} + s_1, \tag{3.25}$$

$$\Pi_2^{(2)} = (\bar{c}_4 - \sigma_2 A_3)\,\mathrm{e}^{k_3(ix+y-R)} + (\sigma_2 A_4 + \bar{c}_3)\,\mathrm{e}^{k_4(ix+y-R)} + s_2, \tag{3.26}$$

where

$$A_q = \sum_{n=1}^{\infty} C_{-n}R^n S_n^{*}(\gamma k_q) \quad (q=1,2), \qquad A_q = \sum_{n=1}^{\infty} C_{n}R^n S_n(\gamma k_q) \quad (q=3,4),$$

$$\sigma_{1,2} = \frac{1+\varepsilon}{(2+\varepsilon)\sqrt{1 \mp 4\overline{\tau}}}$$

and s_1 and s_2 are constants

$$s_1 = C_0 - \frac{K_1}{2\gamma\bar{v}(2+\varepsilon)} + \frac{2G_1(i)}{2+\varepsilon}, \qquad s_2 = C_0 - \frac{K_2}{2\gamma\bar{v}(2+\varepsilon)} + \frac{2F_1(i)}{2+\varepsilon}.$$

From (3.25), (3.26) it is seen that in the moving coordinate system the waves k_1 and k_2 propagate from left to right, and the waves k_3 and k_4, on the contrary, from right to left. The properties of these waves are well studied for homogeneous fluids with a free surface (*e.g.* [1]–[4]) and are completely replicated for a two-layer infinite fluid. The k_2-wave is traveling upstream. The k_1-wave, however, is traveling downstream together with k_3- and k_4-waves. For a submerged body of arbitrary form the potentials of the diffraction waves $\Phi_1^{(2)}$ are

$$\Phi_1^{(2)}(x, y) = \alpha_2 \, \mathrm{e}^{k_2(y - ix)} \quad (x \to \infty), \tag{3.27}$$

$$\Phi_1^{(2)}(x, y) = \alpha_1 \, \mathrm{e}^{k_1(y - ix)} + \alpha_3 \, \mathrm{e}^{k_3(ix + y)} + \alpha_4 \, \mathrm{e}^{k_4(ix + y)} \quad (x \to -\infty), \tag{3.28}$$

where the coefficients α_q $(q = 1, 2, 3, 4)$ are independent of the spatial coordinates and are identified as a result of solving a particular problem (*e.g.* Sturova [10]).

When the submerged body is a circular cylinder, according to the above-mentioned results either the waves with wave numbers k_1 and k_2 exist at infinity, or those with wave numbers k_3 and k_4. They cannot exist at the same time. This result is a generalization of the fact that for $U = 0$ there is no reflection from a submerged circular cylinder (see [1], [4] for details).

In the space-fixed reference frame, there are three types of following waves depending on the forward velocity of the body, whereas the head wave is only the k_4-wave, irrespective of the forward speed. In following waves, when the body speed is less than the group velocity of incident waves $c_g = \omega_0/(2k_0)$, *i.e.* $U < c_g$, the dimensionless wavenumber of the incident wave is equal k_2 . For a body speed greater than the group velocity, but less than the phase velocity of the incident wave $c_p = \omega_0/k_0$, *i.e.* $c_g < U < c_p$, the incident wave is the k_1-wave, and for a body speed higher than the phase velocity $U > c_p$, it is the k_3-wave.

The solutions for each of the possible incoming waves are given below. Let the incoming wave be a k_1-wave. Using (3.27), (3.28) and (2.9), the potential $\Psi^{(2)}$ at the interface in far field, we can present

$$\Psi^{(2)}(x, d) = \mathrm{e}^{-ik_1 x} + \bar{\alpha_2} \, \mathrm{e}^{-ik_2 x} \quad (x \to \infty),$$
$$\Psi^{(2)}(x, d) = (1 + \bar{\alpha_1}) \, \mathrm{e}^{-ik_1 x} \quad (x \to -\infty).$$

In what follows we have

$$\bar{\alpha_q} = -i\sqrt{\frac{k_q}{a\bar{g}}}\alpha_q \, \mathrm{e}^{k_q d} \quad (q = 1, 2, 3, 4).$$

The unknown coefficients C_{-1} and C_{-2} for $\Pi_1^{(2)}$ in (3.24) are identified as a result of fulfilment of the next two conditions in the far field:

(i) the potential of the k_1-wave in (3.25) at $x \to \infty$ is equal to the potential of the incident wave;
(ii) k_2-wave is absent at $x \to -\infty$.

As a result, we obtain a system of two linear equations to determine the vector $\mathbf{B} = (C_{-1}, C_{-2})$

$$\mathbf{AB} = \mathbf{C}, \tag{3.29}$$

where the vector $\mathbf{C} = (\mathrm{e}^{-\gamma k_1}, 0)$ and the matrix $\mathbf{A}$ is

$$\mathbf{A} = \begin{pmatrix} a_2 - \sigma_1 M_{11} & b_2 - \sigma_1 M_{12} \\ a_1 - \sigma_1 M_{21} & b_1 - \sigma_1 M_{22} \end{pmatrix}.$$

Here

$$(M_{11}, M_{12}) = \sum_{n=1}^{\infty} R^n S_n^{+*}(\gamma k_1)(Q_{-n}, T_{-n}),$$

$$(M_{21}, M_{22}) = \sum_{n=1}^{\infty} R^n S_n^{-*}(\gamma k_2)(Q_{-n}, T_{-n}). \tag{3.30}$$

Coefficients Q_{-n} and T_{-n} are computed from the recursive relation (3.8) for C_{-n} at $\mathbf{B} = (1, 0)$ and $\mathbf{B} = (0, 1)$, respectively. On solution of (3.29) we can define all coefficients of the series for $\Pi_1^{(s)}$ in (3.4), (3.5) $C_{-n} = Q_{-n}C_{-1} + T_{-n}C_{-2}$.

It follows from the recursive relation (3.8) that C_{-n} reduce as R^n $(R < 1)$, and from (B6), (B7) that S_n are bounded. Therefore, the series in (3.30) converge as R^{2n}/n, and we can achieve the required accuracy of computations by using a finite number of terms.

For an incoming k_2-wave the potential $\Psi^{(2)}$ in the far field at the interface has the form

$$\Psi^{(2)}(x, d) = (1 + \bar{\alpha}_2)\, \mathrm{e}^{-ik_2x} \quad (x \to \infty),$$

$$\Psi^{(2)}(x, d) = \bar{\alpha}_1\, \mathrm{e}^{-ik_1x} + \mathrm{e}^{-ik_2x} \quad (x \to -\infty).$$

The vector $\mathbf{B}$ is defined with use of the following two conditions:

(i) at $x \to \infty$ the k_1-wave is absent;
(ii) the potential of the k_2-wave at $x \to -\infty$ is equal to the potential of the incoming wave. As a result we get a linear system (3.29), but with the vector $\mathbf{C} = (0, -\mathrm{e}^{-\gamma k_2})$.

For an incoming k_3-wave the potential $\Psi^{(2)}$ in the far field at the interface has the form

$$\Psi^{(2)}(x, d) = \mathrm{e}^{ik_3x} \quad (x \to \infty),$$

$$\Psi^{(2)}(x, d) = (1 + \bar{\alpha}_3)\, \mathrm{e}^{ik_3x} + \bar{\alpha}_4\, \mathrm{e}^{ik_4x} \quad (x \to -\infty). \tag{3.31}$$

Unknown coefficients C_1, C_2 for $\Pi_2^{(2)}$ in (3.24) are determined after the fulfilment of the next two conditions in the far field at $x \to \infty$:

(i) the potential of k_3-wave in (3.26) coincides with the potential of the incoming waves;
(ii) k_4-wave is absent.

We obtain a system of linear equations to determine the vector $\mathbf{D} = (C_1, C_2)$

$$\mathbf{FD} = \mathbf{G}, \tag{3.32}$$

where the vector $\mathbf{G} = (-\mathrm{e}^{-\gamma k_3}, 0)$, and the matrix $\mathbf{F}$ has the form

$$\mathbf{F} = \begin{pmatrix} \sigma_2 M_{31} + a_3 & \sigma_2 M_{32} + b_3 \\ \sigma_2 M_{41} + a_4 & \sigma_2 M_{42} + b_4 \end{pmatrix},$$

$$(M_{31}, M_{32}) = \sum_{n=1}^{\infty} R^n S_n^{+}(\gamma k_3)(V_n, W_n), \qquad (M_{41}, M_{42}) = \sum_{n=1}^{\infty} R^n S_n^{+}(\gamma k_4)(V_n, W_n).$$

Coefficients V_n and W_n are computed according to the recursive relations (3.16) for C_n at $\mathbf{D} = (1, 0)$ and $\mathbf{D} = (0, 1)$, respectively. On solution of (3.32) we can define all coefficients of the series for $\Pi_2^{(s)}$ $C_n = V_n C_1 + W_n C_2$.

For an incoming k_4-wave, we obtain the potential $\Psi^{(2)}$ in the far field on the interface from (3.31) by replacing k_4 with k_3, and $\bar{\alpha_4}$ with $\bar{\alpha_3}$. We get the linear system (3.32), but with the vector $\mathbf{G} = (0, \mathrm{e}^{-\gamma k_4})$.

Upon computation of the vectors $\mathbf{B}$ ($\mathbf{D}$) for waves $k_{1,2}$ ($k_{3,4}$), we can determine completely the diffraction potentials. The coefficient C_0 is determined from the condition $s_1 + s_2 = 0$ (see (3.25), (3.26)), and B_0 does from equation

$$K_1 + K_2 + 2\gamma\bar{\nu}[B_0 - 2C_0(1+\varepsilon)] = 0.$$

This equation is derived on substituting (3.3) in (3.6) and matching the coefficients in front of terms which are independent of θ.

3.2. The exciting forces

Computation of exciting forces (2.11) in dimensionless variables is performed according to the formulas

$$\begin{aligned}\overline{F}_{ej} &= \frac{F_{ej}}{\rho_2\eta_0\bar{g}a} = \sqrt{\frac{2+\varepsilon}{\varepsilon k}}\int_0^{2\pi}\left[i\,\mathrm{Fr}\frac{\partial\Upsilon^{(2)}}{\partial\varphi}\frac{\partial\Psi^{(2)}}{\partial\varphi} - \sqrt{\nu}\Psi^{(2)}\right]n_j\,\mathrm{d}\varphi \\ &= \sqrt{\frac{2+\varepsilon}{\varepsilon k}}(i\,\mathrm{Fr}Y_j - \sqrt{\nu}X_j), \quad (j = 1, 2)\end{aligned}$$

where $\mathrm{Fr} = U/\sqrt{ag}$ is the Froude number and

$$X_j = \int_0^{2\pi}\Psi^{(2)}n_j\,\mathrm{d}\varphi = -\int_0^{2\pi}\Psi^{(2)}\lambda^{-1}n_j\,\mathrm{d}\theta, \tag{3.33}$$

$$Y_j = \int_0^{2\pi}\frac{\partial\Upsilon^{(2)}}{\partial\varphi}\frac{\partial\Psi^{(2)}}{\partial\varphi}n_j\,\mathrm{d}\varphi = -\int_0^{2\pi}\frac{\partial\Upsilon^{(2)}}{\partial\theta}\frac{\partial\Psi^{(2)}}{\partial\theta}\lambda n_j\,\mathrm{d}\theta, \tag{3.34}$$

$$n_1 = -\cos\varphi = \frac{\gamma\cos\theta}{\sin\theta - d}, \qquad n_2 = -\sin\varphi = \frac{d\sin\theta - 1}{d - \sin\theta}, \tag{3.35}$$

$$\lambda(\theta) = \partial\theta/\partial\varphi|_{\rho=1} = (\sin\theta - d)/\gamma. \tag{3.36}$$

The steady potential is given in Appendix A. On the cylinder surface, $\rho = 1$, and using (A1) we have

$$\Upsilon^{(2)} = 2\mathrm{Re}\left\{\sum_{n=1}^{\infty} Z_n\,\mathrm{e}^{in\theta}\right\} = \sum_{n=1}^{\infty}(Z_n\,\mathrm{e}^{in\theta} + Z_n^*\,\mathrm{e}^{-in\theta}). \tag{3.37}$$

The quantities X_j and Y_j are computed separately for the cases of incoming waves with $k = k_{1,2}$ and $k = k_{3,4}$. For incoming waves with $k = k_{1,2}$ on the cylinder surface we have

$$\Psi^{(2)} = 2\sum_{n=1}^{\infty} C_{-n}\,\mathrm{e}^{-in\theta}. \tag{3.38}$$

Substituting (3.37) and (3.38) in (3.33), (3.34), we obtain

$$X_1 = 2\gamma^2 \sum_{n=1}^{\infty} C_{-n} \int_0^{2\pi} \frac{\cos\theta}{(d-\sin\theta)^2}\, \mathrm{e}^{-in\theta}\, \mathrm{d}\theta = -4\pi\gamma \sum_{n=1}^{\infty} (-i)^{n+1} n C_{-n} R^n, \tag{3.39}$$

$$X_2 = 2\gamma \sum_{n=1}^{\infty} C_{-n} \int_0^{2\pi} \frac{1-d\sin\theta}{(d-\sin\theta)^2}\, \mathrm{e}^{-in\theta}\, \mathrm{d}\theta = -4\pi\gamma \sum_{n=1}^{\infty} (-i)^{n} n C_{-n} R^n, \tag{3.40}$$

$$\begin{aligned} Y_1 &= 2\int_0^{2\pi} \sum_{n=1}^{\infty} nC_{-n}\, \mathrm{e}^{-in\theta} \cdot \sum_{n=1}^{\infty} n(Z_n\, \mathrm{e}^{in\theta} - Z_n^*\, \mathrm{e}^{-in\theta}) \cos\theta\, \mathrm{d}\theta \\ &= 2\pi \sum_{n=1}^{\infty} n(n+1)(C_{-n} Z_{n+1} + C_{-n-1} Z_n), \end{aligned}$$

$$\begin{aligned} Y_2 &= \frac{2}{\gamma}\int_0^{2\pi} \sum_{n=1}^{\infty} nC_{-n}\, \mathrm{e}^{-in\theta} \cdot \sum_{n=1}^{\infty} n(Z_n\, \mathrm{e}^{in\theta} - Z_n^*\, \mathrm{e}^{-in\theta})(1 - d\sin\theta)\, \mathrm{d}\theta \\ &= \frac{2\pi}{\gamma} \sum_{n=1}^{\infty} n[2nC_{-n} Z_n - \mathrm{i}d(n+1)(C_{-n} Z_{n+1} - C_{-n-1} Z_n)]. \end{aligned}$$

Evaluations of the integrals in (3.39), (3.40) are presented in [7].

For incoming waves with $k = k_{3,4}$ on the cylinder surface we have

$$\Psi^{(2)} = 2\sum_{n=1}^{\infty} C_n\, \mathrm{e}^{in\theta},$$

$$X_1 = iX_2 = -4\pi\gamma \sum_{n=1}^{\infty} i^{n+1} n C_n R^n, \quad Y_1 = 2\pi \sum_{n=1}^{\infty} n(n+1)(C_n Z_{n+1}^* + C_{n+1} Z_n^*),$$

$$Y_2 = \frac{2\pi}{\gamma} \sum_{n=1}^{\infty} n[2nC_n Z_n^* + \mathrm{i}d(n+1)(C_n Z_{n+1}^* - C_{n+1} Z_n^*)].$$

The series arising in the computation of the exciting forces converge as R^{2n} $(R < 1)$.

In Tables 1, 2 we give results for $\overline{F}_{ej}$ on the cylinder submerged at $h = 2a$ and for $\mathrm{Fr} = 0{\cdot}6$ (homogeneous fluid with free surface, *i.e.* $\varepsilon \to \infty$) and $\overline{\mathrm{Fr}} = U/\sqrt{\bar{g}a} = 0{\cdot}6$ (two-layer fluid with $\varepsilon = 0{\cdot}03$), respectively. This value of the Froude number was chosen because, within the range $k_0 a \leqslant 3$, there are all possible incoming waves. For the wave coming from the right, the critical point exists at $k_0 a \simeq 0{\cdot}11915$ $\quad (\bar{\tau} = \frac{1}{4})$. For the wave coming from the left, the behaviour of the exciting forces is more complex. When k_0 is small, we have $k_0 a = k_2$. As the wave frequency increases, k_0 reaches the critical point at $k_0 a \simeq 0{\cdot}69444$ that correspond $U = c_g$. The amplitudes of the forces vanish and the phase difference changes significantly at this critical wavenumber. But the problem will not become supercritical as k_0 increases further. This is because ω in Equation (2.1) will decrease when $k_0 a > 0{\cdot}69444$ and the flow is again subcritical, but with $k_0 a = k_1$. When $k_0 a \simeq 2{\cdot}77778$, we have $\omega = 0 \quad (U = c_p)$. At this wavenumber the real parts of both exciting forces have discontinuities at which the absolute values are continuous, but the signs change. At greater wavenumbers the incident wave is coming from the left in the moving system, and we have $k_0 a = k_3$. Only when $k_0 a \simeq 4{\cdot}04752$ ($\bar{\tau} = \frac{1}{4}$), does the flow become supercritical. However, this has no effect on the submerged circular cylinder.

Similar results for a homogeneous fluid with free surface at Fr $= 0{\cdot}4$ are given in [4]. Results are only listed up to $k_0 a = 1$ with step 0·05. The exciting forces were normalized as $\pi \bar{F}_{ej}\, \mathrm{e}^{k_0 h}/(a k_0)$. Comparison of numerical results [4] with results of our solution shows a relative difference of less than 1%. Furthermore, in [2], [4] it was pointed out that the exciting forces are continuous at the critical point $\tau = \frac{1}{4}$. Our results confirm this statement for the two-layer infinite fluid at $\bar{\tau} = \frac{1}{4}$.

4. A circular cylinder submerged in the upper layer

Having performed the change $y = y_1 - d$, let us transfer the origin of coordinates to the cylinder center. The solution of this problem repeats in many aspects the reasonings of Section 3 and, therefore, will be given here in brief.

Instead of (3.1), the conformal mapping is now given by

$$w = \frac{i + Rz}{R - iz}. \tag{4.1}$$

The upper layer is contained in the annular region $R < |w| < 1$, the lower layer is contained in the circular region $|w| < R$.

The solution of the steady problem (2.3)–(2.6) is described in Appendix A.

4.1. The diffraction potentials

The potentials $\Psi^{(s)}$ are presented as the series

$$\Psi^{(1)} = \sum_{n=-\infty}^{\infty} C_n (\rho^n + \rho^{-n})\, \mathrm{e}^{in\theta}, \quad \Psi^{(2)} = \sum_{n=-\infty}^{\infty} B_n \rho^{|n|}\, \mathrm{e}^{in\theta}.$$

Results obtained in Section 3 demonstrate that the constants B_0 and C_0 can be taken zero for calculation of the exciting forces. Using the kinematic boundary condition in (2.8), we express the coefficients B_n through C_n and write the potentials in the form (3.3), where now

$$\Pi_1^{(1)}(\rho, \theta) = \sum_{n=1}^{\infty} C_{-n} (\rho^n + \rho^{-n})\, \mathrm{e}^{-in\theta}, \qquad \Pi_2^{(1)}(\rho, \theta) = \sum_{n=1}^{\infty} C_n (\rho^n + \rho^{-n})\, \mathrm{e}^{in\theta}, \tag{4.2}$$

$$\Pi_1^{(2)}(\rho, \theta) = \sum_{n=1}^{\infty} C_{-n} R_n^- R^{-n} \rho^n\, \mathrm{e}^{-in\theta}, \quad \Pi_2^{(2)}(\rho, \theta) = \sum_{n=1}^{\infty} C_n R_n^- R^{-n} \rho^n\, \mathrm{e}^{in\theta}. \tag{4.3}$$

In the variables ρ, θ the dynamic condition at the interface has the form (3.6) as before, but now in (3.7)

$$\frac{\partial \rho}{\partial y} = -R \frac{\partial \theta}{\partial x} = \frac{R}{\gamma}(1 - \sin\theta). \tag{4.4}$$

By substituting the representation for $\Psi^{(1,2)}$ as a sum (3.3) of the series (4.2), (4.3) in (3.6) and using (4.4), we obtain a recursive relation for the series coefficients.

We consider the solution for $\Pi_1^{(1,2)}$ (*case 1*) and $\Pi_2^{(1,2)}$ (*case 2*) in its own right.

(a) *Case 1*. The recursive relations for C_{-n} are

$$
\begin{aligned}
& C_{2-n}\frac{\tau^2}{2\nu\gamma}(n-2)(n-1)\tilde{P}_{n-2} \\
& \quad + C_{1-n}i(n-1)\left\{\left[\frac{\tau}{\nu\gamma}(2n-1)-2\right]\tau\tilde{P}_{n-1}-\varepsilon R_{n-1}^{-}\right\} \\
& \quad + C_{-n}\left[\left(4n\tau-2\nu\gamma-\frac{3n^2\tau^2}{\nu\gamma}\right)\tilde{P}_n+2\varepsilon n R_n^{-}\right] \\
& \quad + C_{-n-1}i(n+1)\left\{\left[2-\frac{\tau}{\nu\gamma}(2n+1)\right]\tau\tilde{P}_{n+1}+\varepsilon R_{n+1}^{-}\right\} \\
& \quad + C_{-n-2}\frac{\tau^2}{2\nu\gamma}(n+2)(n+1)\tilde{P}_{n+2}=0,
\end{aligned}
$$

where

$$\tilde{P}_n = \varepsilon R^n - (2+\varepsilon)R^{-n}.$$

Once the coefficients C_{-1} and C_{-2} have been defined, the series for $\Pi_1^{(1,2)}$ are fully determined. We introduce additional functions $G_1(\xi)$ and $G_2(\xi)$ as in (3.9). For their sum with weights

$$\tilde{W}_1(\xi) = \varepsilon G_1(\xi) - (2+\varepsilon)G_2(\xi) \tag{4.5}$$

we obtain a differential equation

$$
\begin{aligned}
& \frac{\bar{\tau}^2}{2\bar{\nu}\gamma}(1-i\xi)^4\tilde{W}_1'' - i(1-i\xi)^2\left[\frac{\bar{\tau}^2}{\bar{\nu}\gamma}(1-i\xi)-2\bar{\tau}-1\right]\tilde{W}_1' - 2\bar{\nu}\gamma\tilde{W}_1 \\
& \quad = \tilde{K}_1 - 2i(1-i\xi)^2 G_1',
\end{aligned} \tag{4.6}
$$

where

$$\tilde{K}_1 = \frac{\bar{\tau}^2}{\bar{\nu}\gamma}C_{-2}\tilde{P}_2 - iC_{-1}\left[\left(\frac{\bar{\tau}}{\bar{\nu}\gamma}-2\right)\bar{\tau}\tilde{P}_1-(2+\varepsilon)R_1^{-}\right]. \tag{4.7}$$

The general solution of (4.6) is

$$
\begin{aligned}
\tilde{W}_1 \;=\; & \frac{\bar{\nu}}{\bar{\tau}^2(k_3-k_4)}\sum_{n=1}^{\infty}C_{-n}R^n[I_n(\gamma k_4,\xi)-I_n(\gamma k_3,\xi)] - c_1\exp\left(-\frac{2\gamma k_4}{1-i\xi}\right) \\
& + c_2\exp\left(-\frac{2\gamma k_3}{1-i\xi}\right) - \frac{\tilde{K}_1}{2\gamma\bar{\nu}},
\end{aligned}
$$

where I_n is defined in (3.14), $k_{3,4}$ are given in (3.21), and $c_{1,2}$ are constants of integration

$$(c_1, c_2) = (2+\varepsilon)(\tilde{c}_1\,\mathrm{e}^{2\gamma k_4}, \tilde{c}_2\,\mathrm{e}^{2\gamma k_3}),$$

$$\tilde{c}_1 = \tilde{a}_3 C_{-1} + \tilde{b}_3 C_{-2}, \quad \tilde{c}_2 = \tilde{a}_4 C_{-1} + \tilde{b}_4 C_{-2},$$

$$\tilde{a}_q = \frac{1}{(2+\varepsilon)(k_3-k_4)} \left(\frac{i\tilde{P}_1}{2\gamma} - \frac{\tilde{v}_1 k_q}{\bar{\nu}} \right), \qquad \tilde{b}_q = -\frac{k_q \tilde{v}_2}{\bar{\nu}(2+\varepsilon)(k_3-k_4)} \quad (q = 3, 4),$$

$$\tilde{v}_1 = -\frac{i}{2\gamma} \left[\left(\frac{\bar{\tau}}{\bar{\nu}\gamma} - 2 \right) \bar{\tau} \tilde{P}_1 - (2+\varepsilon) R_1^- \right], \qquad \tilde{v}_2 = \frac{\bar{\tau}^2}{2\bar{\nu}\gamma^2} \tilde{P}_2. \tag{4.8}$$

(b) *Case 2.*
The solution for $\Pi_2^{(1,2)}$ is derived in a similar way. The recursive relations for C_n are

$$\begin{aligned}
&C_{n-2} \frac{\tau^2}{2\nu\gamma} (n-2)(n-1) \tilde{P}_{n-2} \\
&\quad - C_{n-1} i(n-1) \left\{ \left[2 + \frac{\tau}{\nu\gamma}(2n-1) \right] \tau \tilde{P}_{n-1} - \varepsilon R_{n-1}^- \right\} \\
&\quad - C_n \left[\left(4n\tau + 2\nu\gamma + \frac{3n^2\tau^2}{\nu\gamma} \right) \tilde{P}_n - 2\varepsilon n R_n^- \right] \\
&\quad + C_{n+1} i(n+1) \left\{ \left[\frac{\tau(2n+1)}{\nu\gamma} + 2 \right] \tau \tilde{P}_{n+1} - \varepsilon R_{n+1}^- \right\} \\
&\quad + C_{n+2} \frac{\tau^2}{2\nu\gamma} (n+2)(n+1) \tilde{P}_{n+2} = 0,
\end{aligned} \tag{4.9}$$

Once C_1 and C_2 have been defined, the series for $\Pi_2^{(1,2)}$ are completely determined due to (4.9). To define C_1 and C_2 we introduce additional functions $F_1(\xi)$ and $F_2(\xi)$ as in (3.17). For their sum with weights

$$\tilde{W}_2(\xi) = \varepsilon F_1(\xi) - (2+\varepsilon) F_2(\xi) \tag{4.10}$$

we obtain a differential equation

$$\begin{aligned}
&\frac{\bar{\tau}^2}{2\bar{\nu}\gamma} (1+i\xi)^4 \tilde{W}_2'' + i(1+i\xi)^2 \left[\frac{\bar{\tau}^2}{\bar{\nu}\gamma}(1+i\xi) + 2\bar{\tau} - 1 \right] \tilde{W}_2' - 2\bar{\nu}\gamma \tilde{W}_2 \\
&\quad = \tilde{K}_2 + 2i(1+i\xi)^2 F_1',
\end{aligned} \tag{4.11}$$

where

$$\tilde{K}_2 = \frac{\bar{\tau}^2}{\bar{\nu}\gamma} C_2 \tilde{P}_2 + i C_1 \left[\left(\frac{\bar{\tau}}{\bar{\nu}\gamma} + 2 \right) \bar{\tau} \tilde{P}_1 - (2+\varepsilon) R_1^- \right].$$

The solution of Equation (4.11) is

$$\begin{aligned}
\tilde{W}_2 \;=\; &\frac{\bar{\nu}}{\bar{\tau}^2(k_1-k_2)} \sum_{n=1}^{\infty} C_n R^n [J_n(\gamma k_1, \xi) - J_n(\gamma k_2, \xi)] + c_3 \exp\left(-\frac{2\gamma k_2}{1+i\xi} \right) \\
&+ c_4 \exp\left(-\frac{2\gamma k_1}{1+i\xi} \right) - \frac{\tilde{K}_2}{2\gamma\bar{\nu}},
\end{aligned}$$

where J_n is defined in (3.22), $k_{1,2}$ are given in (3.13), and $c_{3,4}$ are the constants of integration

$$(c_3, c_4) = (2+\varepsilon)(\tilde{c}_3\, \mathrm{e}^{2\gamma k_2}, \tilde{c}_4\, \mathrm{e}^{2\gamma k_1}),$$

$$\tilde{c}_3 = \tilde{a}_1 C_1 + \tilde{b}_1 C_2, \quad \tilde{c}_4 = -(\tilde{a}_2 C_1 + \tilde{b}_2 C_2),$$

$$\tilde{a}_q = \frac{1}{(2+\varepsilon)(k_1-k_2)}\left(\frac{i\tilde{P}_1}{2\gamma} + \frac{\tilde{v}_3 k_q}{\bar{v}}\right), \qquad \tilde{b}_q = \frac{k_q \tilde{v}_2}{\bar{v}(2+\varepsilon)(k_1-k_2)} \quad (q = 1, 2),$$

$$\tilde{v}_3 = \frac{i}{2\gamma}\left[\left(\frac{\bar{\tau}}{\bar{v}\gamma} + 2\right)\bar{\tau}\tilde{P}_1 - (2+\varepsilon)R_1^-\right],$$

$\tilde{v}_2$ is defined in (4.8).

Let $\xi = R\rho^{-1}\,\mathrm{e}^{-i\theta}$ for $\tilde{W}_1(\xi)$ in (4.5) and $\xi = R\rho^{-1}\,\mathrm{e}^{i\theta}$ for $\tilde{W}_2(\xi)$ in (4.10). Then similar to (3.23) we can write

$$\tilde{W}_1 = \sum_{n=1}^{\infty} C_{-n}[\varepsilon(R^{2n}-1)-2]\rho^{-n}\,\mathrm{e}^{-in\theta},$$

$$\tilde{W}_2 = \sum_{n=1}^{\infty} C_n[\varepsilon(R^{2n}-1)-2]\rho^{-n}\,\mathrm{e}^{in\theta}. \tag{4.12}$$

Let us consider a behaviour of the upper layer potential $\Psi^{(1)}$ at the interface. According to (4.2) and (4.3) at $\rho = R$ we get

$$\Pi_1^{(1)}(R,\theta) = \sum_{n=1}^{\infty} C_{-n} R_n^+\,\mathrm{e}^{-in\theta}, \quad \Pi_2^{(1)}(R,\theta) = \sum_{n=1}^{\infty} C_n R_n^+\,\mathrm{e}^{in\theta}.$$

Using (4.12) we have

$$\Pi_1^{(1)}(R,\theta) = [2(1+\varepsilon)\tilde{G}_1 - \tilde{W}_1]/(2+\varepsilon),$$
$$\Pi_2^{(1)}(R,\theta) = [2(1+\varepsilon)\tilde{F}_1 - \tilde{W}_2]/(2+\varepsilon).$$

By applying the properties of integral functions $J_n(\beta, \xi)$ (see Appendix B), we determine the behaviour of $\Pi_{1,2}^{(1)}$ at $\theta \to \pi/2$ ($\xi \to i$), which is equivalent to $|x| \to \infty$. As a result the potentials $\Pi_{1,2}^{(1)}$ at $|x| \to \infty$ can be represented in the form

$$\Pi_1^{(1)} = (\tilde{\sigma}_1 \tilde{A}_4 + \tilde{c}_1)\,\mathrm{e}^{k_4(ix-y-R)} - (\tilde{\sigma}_1 \tilde{A}_3 + \tilde{c}_2)\,\mathrm{e}^{k_3(ix-y-R)},$$

$$\Pi_2^{(1)} = (\tilde{\sigma}_2 \tilde{A}_2)\,\mathrm{e}^{-k_2(ix+y+R)} - (\tilde{\sigma}_2 \tilde{A}_1 + \tilde{c}_4)\,\mathrm{e}^{-k_1(ix+y+R)},$$

where

$$\tilde{A}_q = \sum_{n=1}^{\infty} C_n R^n S_n(\gamma k_q) \quad (q = 1, 2), \qquad \tilde{A}_q = \sum_{n=1}^{\infty} C_{-n} R^n S_n^*(\gamma k_q) \quad (q = 3, 4),$$

$$\tilde{\sigma}_{1,2} = \frac{1}{(2+\varepsilon)\sqrt{1 \pm 4\bar{\tau}}}.$$

Derivation of a final solution for each of the possible incoming waves is similar to that in Section 3.1. The condition in the far field from which the coefficients $C_{1,2}$ are determined for waves $k_{1,2}$ and $C_{-1,-2}$ for waves $k_{3,4}$ are the same those as given in Section 3.1 for a relevant wave k_q $(q = 1, 2, 3, 4)$.

4.2. The exciting forces

Computation of exciting forces (2.11) in dimensionless variables is performed according to formulae

$$\tilde{F}_{ej} = \frac{F_{ej}}{\rho_1 \eta_0 \bar{g} a} = \sqrt{\frac{2+\varepsilon}{\varepsilon k}} \int_0^{2\pi} \left[\sqrt{\nu}\Psi^{(1)} - i\,\mathrm{Fr}\frac{\partial \Upsilon^{(1)}}{\partial \varphi}\frac{\partial \Psi^{(1)}}{\partial \varphi} \right] n_j\,\mathrm{d}\varphi$$

$$= \sqrt{\frac{2+\varepsilon}{\varepsilon k}} (\sqrt{\nu}\tilde{X}_j - i\,\mathrm{Fr}\tilde{Y}_j) \quad (j = 1, 2)$$

where

$$\tilde{X}_j = \int_0^{2\pi} \Psi^{(1)} n_j\,\mathrm{d}\varphi = -\int_0^{2\pi} \Psi^{(1)} \lambda^{-1} n_j\,\mathrm{d}\theta,$$

$$\tilde{Y}_j = \int_0^{2\pi} \frac{\partial \Upsilon^{(1)}}{\partial \varphi}\frac{\partial \Psi^{(1)}}{\partial \varphi} n_j\,\mathrm{d}\varphi = -\int_0^{2\pi} \frac{\partial \Upsilon^{(1)}}{\partial \theta}\frac{\partial \Psi^{(1)}}{\partial \theta} \lambda n_j\,\mathrm{d}\theta,$$

$$n_1 = \gamma \cos\theta/(d - \sin\theta), \qquad n_2 = (1 - d\sin\theta)/(d - \sin\theta).$$

The expression for λ is presented in (3.36).

The solution for the steady potential $\Upsilon^{(1)}$ is given in Appendix A and on the cylinder surface at $\rho = 1$ by use of (A12) is

$$\Upsilon^{(1)} = 2\mathrm{Re}\left\{ \sum_{n=1}^{\infty} Z_n\,\mathrm{e}^{in\theta} \right\} = \sum_{n=1}^{\infty} (Z_n\,\mathrm{e}^{in\theta} + Z_n^*\,\mathrm{e}^{-in\theta}). \tag{4.13}$$

For incoming waves with $k = k_{1,2}$ we obtain

$$\tilde{X}_1 = i\tilde{X}_2 = 4\pi\gamma \sum_{n=1}^{\infty} i^{n+1} n C_n R^n, \qquad \tilde{Y}_1 = -2\pi \sum_{n=1}^{\infty} n(n+1)(C_n Z_{n+1}^* + C_{n+1} Z_n^*),$$

$$\tilde{Y}_2 = -\frac{2\pi}{\gamma} \sum_{n=1}^{\infty} n[2nC_n Z_n^* + \mathrm{i}d(n+1)(C_n Z_{n+1}^* - C_{n+1} Z_n^*)].$$

For incoming waves with $k = k_{3,4}$

$$\tilde{X}_1 = -i\tilde{X}_2 = 4\pi\gamma \sum_{n=1}^{\infty} (-i)^{n+1} n C_{-n} R^n,$$

$$\tilde{Y}_1 = -2\pi \sum_{n=1}^{\infty} n(n+1)(C_{-n} Z_{n+1} + C_{-n-1} Z_n),$$

Table 1. Exciting forces on a cylinder under a free surface in a homogeneous fluid with $h = 2a$ and $\mathrm{Fr} = 0{\cdot}6$.

	Wave from the left				Wave from the right			
k_0a	$\mathrm{Re}(\bar{F}_{e1})$	$\mathrm{Im}(\bar{F}_{e1})$	$\mathrm{Re}(\bar{F}_{e2})$	$\mathrm{Im}(\bar{F}_{e2})$	$\mathrm{Re}(\bar{F}_{e1})$	$\mathrm{Im}(\bar{F}_{e1})$	$\mathrm{Re}(\bar{F}_{e2})$	$\mathrm{Im}(\bar{F}_{e2})$
0·1	−0·25637	−0·94515	1·18401	−0·29152	0·02296	0·58647	0·42760	−0·01056
0·2	−1·46555	−1·13977	1·38610	−1·69758	0·07849	0·92148	0·73267	−0·05413
0·3	−1·44796	−0·09611	0·15285	−1·66927	0·12495	1·07288	0·88160	−0·09345
0·4	−0·84911	0·14583	−0·14098	−0·97405	0·14468	1·11097	0·92840	−0·11135
0·6	−0·18382	0·05175	−0·05261	−0·20963	0·12835	1·02643	0·87063	−0·09949
0·8	−0·16072	−0·04384	0·04695	−0·18186	0·08929	0·86569	0·73702	−0·06729
1·0	−0·39600	−0·11427	0·12330	−0·44947	0·05554	0·69575	0·59118	−0·03928
1·4	−0·58931	−0·51450	0·58497	−0·67586	0·01819	0·41071	0·34393	−0·00908
1·8	−0·08601	−0·46135	0·52989	−0·09416	0·00529	0·22307	0·18192	0·00021
2·2	−0·01065	−0·22394	0·25740	−0·00841	0·00140	0·11378	0·08913	0·00209
2·6	−0·00201	−0·11445	0·13217	0·00012	0·00028	0·05498	0·04053	0·00197
3·0	0·00034	−0·05941	−0·06912	0·00112	−0·00004	0·02519	0·01686	0·00147

Table 2. Exciting forces on a cylinder under a lower fluid with $h = 2a$ and $\overline{\mathrm{Fr}} = 0{\cdot}6$ for a two-layer fluid with $\varepsilon = 0{\cdot}03$.

	Wave from the left				Wave from the right			
k_0a	$\mathrm{Re}(\bar{F}_{e1})$	$\mathrm{Im}(\bar{F}_{e1})$	$\mathrm{Re}(\bar{F}_{e2})$	$\mathrm{Im}(\bar{F}_{e2})$	$\mathrm{Re}(\bar{F}_{e1})$	$\mathrm{Im}(\bar{F}_{e1})$	$\mathrm{Re}(\bar{F}_{e2})$	$\mathrm{Im}(\bar{F}_{e2})$
0·1	−0·08339	−0·72116	0·86090	−0·09020	0·01111	0·56803	0·45256	−0·00589
0·2	−0·47209	−1·26927	1·45430	−0·52292	0·03925	0·91295	0·77453	−0·02934
0·3	−1·08303	−1·19839	1·35697	−1·19792	0·06528	1·09373	0·95123	−0·05232
0·4	−1·19585	−0·57655	0·65623	−1·31912	0·07894	1·16386	1·02576	−0·06491
0·6	−0·35321	0·00566	−0·00102	−0·38854	0·07488	1·12141	1·00145	−0·06220
0·8	−0·30473	−0·08615	0·09255	−0·33452	0·05425	0·97334	0·87385	−0·04431
1·0	−0·62632	−0·39766	0·43274	−0·68837	0·03456	0·79950	0·71892	−0·02708
1·4	−0·24286	−0·71281	0·78291	−0·26644	0·01159	0·48983	0·43916	−0·00729
1·8	−0·02959	−0·39554	0·43520	−0·03037	0·00342	0·27680	0·24615	−0·00078
2·2	−0·00463	−0·20586	0·22702	−0·00338	0·00093	0·14829	0·13024	0·00076
2·6	−0·00093	−0·10798	0·11961	0·00009	0·00021	0·07639	0·06597	0·00088
3·0	0·00017	−0·05636	−0·06281	0·00053	0·00001	0·03815	0·03223	0·00069

$$\tilde{Y}_2 = -\frac{2\pi}{\gamma}\sum_{n=1}^{\infty} n[2nC_{-n}Z_n - \mathrm{i}d(n+1)(C_{-n}Z_{n+1} - C_{-n-1}Z_n)].$$

In Table 3 we give results for $\tilde{F}_{ej}$ on the cylinder located above the interface at $h = 2a$ and for $\overline{\mathrm{Fr}} = 0{\cdot}6$ with $\varepsilon = 0{\cdot}03$. Critical values of k_0a fully correspond to those existing for results given in Tables 1 and 2 (see Section 3.2). Using the results given in Tables 1–3, we can calculate both the amplitudes and the phase differences of exciting forces. The qualitative behaviours of the amplitudes of exciting forces are similar to each other in the all

Table 3. Exciting forces on a cylinder in upper fluid with $h = 2a$ and $\overline{\mathrm{Fr}} = 0{\cdot}6$ for a two-layer fluid with $\varepsilon = 0{\cdot}03$.

	Wave from the left				Wave from the right			
k_0a	Re($\tilde{F}_{e1}$)	Im($\tilde{F}_{e1}$)	Re($\tilde{F}_{e2}$)	Im($\tilde{F}_{e2}$)	Re($\tilde{F}_{e1}$)	Im($\tilde{F}_{e1}$)	Re($\tilde{F}_{e2}$)	Im($\tilde{F}_{e2}$)
0·1	−0·07995	−0·71536	−0·85275	0·08636	0·01077	0·56749	−0·45328	0·00573
0·2	−0·45010	−1·26055	−1·44237	0·49792	0·03809	0·91266	−0·77572	0·02854
0·3	−1·04698	−1·21999	−1·37897	1·15666	0·06343	1·09428	−0·95324	0·05094
0·4	−1·19190	−0·61678	−0·69968	1·31328	0·07680	1·16538	−1·02865	0·06327
0·6	−0·36290	0·00143	−0·00359	0·39878	0·07300	1·12435	−1·00550	0·06076
0·8	−0·31295	−0·08996	−0·09666	0·34322	0·05296	0·97677	−0·87820	0·04335
1·0	−0·63111	−0·41712	−0·45373	0·69294	0·03376	0·80286	−0·72302	0·02652
1·4	−0·23212	−0·71169	−0·78078	0·25430	0·01133	0·49241	−0·44224	0·00716
1·8	−0·02843	−0·39378	−0·43272	0·02912	0·00334	0·27854	−0·24821	0·00078
2·2	−0·00448	−0·20536	−0·22619	0·00326	0·00091	0·14940	−0·13155	−0·00073
2·6	−0·00091	−0·10780	−0·11925	−0·00009	0·00021	0·07708	−0·06678	−0·00085
3·0	0·00016	−0·05627	0·06262	−0·00051	0·00001	0·03857	−0·03272	−0·00067

considered cases. However, the phase difference of about 180 degrees exists between heave exciting forces on cylinder located in the upper and the lower layers.

5. Discussion

The explicit solution, which was obtained in [5] for surface-wave diffraction by a circular cylinder without forward speed and then extended in [7], [8] for case of a two-layer fluid, is presented in this work for the general case of internal wave diffraction in a uniform current of a two-layer fluid. The solution is obtained in the form of rapidly converging series and makes it possible to investigate relatively easily all the characteristics of the flow depending on the parameters of the problem. The numerical results presented in Tables 1–4 are obtained from only the nine first terms in the series. Further increase of the number of terms does not change the results. The problem considered here is one constituent of the linear theory of seakeeping. We think that the solution of the seakeeping problem for a body of an arbitrary form is possible only with numerical methods. There arises the question about the estimation of the accuracy of the numerical algorithms used. For this purpose, it is useful to have test solutions of similar problems for bodies of simple geometry. Usually, for 2-D flows a circular cylinder is such a body. For a two-layer fluid an effective numerical method of the solution of a linear problem of seakeeping for a submerged body is the coupled finite-element method (CFEM). This method was applied for steady flow by Sturova [11] and for radiation and diffraction of internal waves by a submerged cylinder at forward speed by Sturova [10]. In the first paper, the upper layer can be bounded by a rigid lid or free surface, in the second one by only a rigid lid. The comparison of the numerical results obtained for a circular cylinder in an unbounded two-layer fluid by CFEM and the given explicit solution showed fair agreement.

Appendix A. The steady problem

The solution technique for the steady problem (2.3)–(2.6) is a multilateral replica of that used in Sections 3, 4 for a diffraction problem appropriate to a cylinder located in the lower and upper layer.

A.1. *The cylinder in the lower layer*

In dimensionless variables, using the conformal mapping (3.1), we write the potentials $\Upsilon^{(s)}$ as the series

$$\Upsilon^{(1)} = \mathrm{Re}\left\{\sum_{n=1}^{\infty} Z_n R_n^- R^{-n} \rho^n \,\mathrm{e}^{in\theta}\right\}, \qquad \Upsilon^{(2)} = \mathrm{Re}\left\{\sum_{n=1}^{\infty} Z_n (\rho^n + \rho^{-n}) \,\mathrm{e}^{in\theta}\right\}. \tag{A1}$$

The notation is identical to that of Section 3.

In the coordinates ρ, θ the dynamic condition at the interface (2.6) has the form

$$\frac{\partial^2 \Upsilon_\varepsilon}{\partial \theta^2}\left(\frac{\partial \theta}{\partial x}\right)^2 + \frac{\partial \Upsilon_\varepsilon}{\partial \theta}\frac{\partial^2 \theta}{\partial x^2} + \varepsilon\mu \frac{\partial \Upsilon^{(2)}}{\partial \rho}\frac{\partial \rho}{\partial y} = 0 \quad (\rho = R),$$

where

$$\Upsilon_\varepsilon = (1+\varepsilon)\Upsilon^{(2)} - \Upsilon^{(1)}, \quad \mu = ag/U^2 = 1/\mathrm{Fr}^2.$$

The recursive relations for the coefficients in (A1) are

$$\begin{aligned} &Z_{n-2}(n-2)(n-1)P_{n-2} - 2Z_{n-1} i(n-1)[(2n-1)P_{n-1} + \varepsilon\mu\gamma R_{n-1}^-] \\ &\quad -2Z_n n(3nP_n + 2\varepsilon\mu\gamma R_n^-) + 2Z_{n+1} i(n+1)[(2n+1)P_{n+1} + \varepsilon\mu\gamma R_{n+1}^-] \\ &\quad +Z_{n+2}(n+2)(n+1)P_{n+2} = 0. \end{aligned} \tag{A2}$$

The coefficients Z_1 and Z_2 are determined from the radiation condition from which, far ahead of the cylinder, the potentials $\Upsilon^{(s)}$ correspond the a uniform current. It is convenient to introduce new functions

$$\bar{F}_1(\xi) = \sum_{n=1}^{\infty} Z_n R^n \xi^n, \qquad \bar{F}_2(\xi) = \sum_{n=1}^{\infty} Z_n R^{-n} \xi^n, \tag{A3}$$

which are similar to those used in (3.17). For their sum with weights

$$\bar{W}_1(\xi) = \varepsilon \bar{F}_1(\xi) + (2+\varepsilon)\bar{F}_2(\xi) \tag{A4}$$

we obtain a differential equation

$$(1+i\xi)^4 \bar{W}_1'' + 2i(1+i\xi)^2(1+i\xi - \bar{\mu}\gamma)\bar{W}_1' = K_0 - 4i\bar{\mu}\gamma(1+\varepsilon)(1+i\xi)^2 \bar{F}_1',$$

where

$$\bar{\mu} = \varepsilon\mu/(2+\varepsilon), \quad K_0 = 2\{Z_2 P_2 + iZ_1[P_1 - 2\gamma^2\bar{\mu}(1+\varepsilon)]\}.$$

The general solution of this equation is

$$\bar{W}_1' = \frac{1}{(1+i\xi)^2}\left\{\frac{iK_0}{2\beta}\left(1-\exp\frac{2i\beta\xi}{1+i\xi}\right) - 2i\beta(1+\varepsilon)\left[2\bar{F}_1 + \sum_{n=1}^{\infty} Z_n R^n J_n(\beta,\xi)\right]\right.$$
$$\left. + c_0 \exp\frac{-2\beta}{1+i\xi}\right\},$$

where $\beta = \gamma\bar{\mu}$, c_0 is the constant of integration, J_n is defined in (3.22). Below, the derivative of $\bar{W}_1$ is used only. From (A4) $\bar{W}_1'(0) = Z_1 P_1$ and we have $c_0 = Z_1 P_1\,\mathrm{e}^{2\beta}$.

Let us consider the behaviour of the x-derivative of $\Upsilon^{(2)}$ at the interface

$$\frac{\partial\Upsilon^{(2)}}{\partial x}\Big|_{y=d} = \frac{\partial\Upsilon^{(2)}}{\partial\theta}\frac{\partial\theta}{\partial x}\Big|_{\rho=R} = \frac{(1+i\,\mathrm{e}^{i\theta})^2}{2\gamma}\sum_{n=1}^{\infty} n Z_n R_n^+\,\mathrm{e}^{i(n-1)\theta}$$
$$= \frac{(1+i\,\mathrm{e}^{i\theta})^2}{2\gamma(2+\varepsilon)}(\bar{W}_1' + 2\bar{F}_1'). \tag{A5}$$

The latter equality in (A5) follows from the form of $\bar{W}_1'$ at $\xi = \mathrm{e}^{i\theta}$ according to (A4). Using the results of Appendix B, we obtain for $|x| \to \infty$

$$\frac{\partial\Upsilon^{(2)}}{\partial x} = \frac{1}{2\gamma(2+\varepsilon)}\left\{\frac{iK_0}{2\beta} + \exp\bar{\mu}(ix+y-R)\right.$$
$$\left.\times\left[Z_1P_1 - \frac{iK_0}{2\beta} - 2i\beta(1+\varepsilon)\sum_{n=1}^{\infty} Z_n R^n S_n(\beta)\right]\right\}. \tag{A6}$$

From the radiation condition the wave motion can exist only downstream of the cylinder, *i.e.*

$$\partial\Upsilon^{(2)}/\partial x \to -1 \quad (x\to\infty). \tag{A7}$$

This means that the expression in square brackets in (A6) for $x \to \infty$ must be zero. Using (A7) for the remaining non-wave part, we have the following linear second-order system for the definition of Z_1 and Z_2

$$Z_1\left[2i\beta(1+\varepsilon)\sum_{n=1}^{\infty} A_n R^n S_n^+(\beta) - P_1\right]$$
$$+ Z_2 2i\beta(1+\varepsilon)\sum_{n=1}^{\infty} B_n R^n S_n^+(\beta) = 2\gamma(2+\varepsilon), \tag{A8}$$

$$Z_1[P_1 - 2\gamma\beta(2+\varepsilon)] - Z_2 i P_2 = 2\gamma\beta(2+\varepsilon). \tag{A9}$$

Here A_n and B_n are computed from the recursive relations (A2) at $Z_1 = 1,\ Z_2 = 0$ and $Z_1 = 0,\ Z_2 = 1$, respectively. On solving (A8), (A9), we can define all coefficients Z_n of the series (A1)

$$Z_n = A_n Z_1 + B_n Z_2. \tag{A10}$$

In the limiting cases $\bar{\mu} \to 0$ and $\bar{\mu} \to \infty$ the problem can be essentially simplified. At $\bar{\mu} \to 0$ ($\overline{\mathrm{Fr}} \to \infty$) we have the problem of the uniform current of a weightless two-layer fluid past a cylinder. The assumption $\bar{\mu} \to \infty$ ($\overline{\mathrm{Fr}} \to 0$) is equivalent the change of the interface by the solid plane. In both limiting cases to find the coefficients Z_n in the recursive relations (A2) we need to determine Z_1 only (see [8] for more details).

Computation of the steady loads (2.10) is performed in the following way

$$\bar{F}_{sj} = \frac{F_{sj}}{\rho_2 a U^2} = \frac{1}{2}\int_0^{2\pi} |\nabla\Upsilon^{(2)}|^2 n_j \, \mathrm{d}\varphi = -\frac{1}{2}\int_0^{2\pi} \left(\frac{\partial \Upsilon^{(2)}}{\partial\theta}\right)^2 \lambda n_j \, \mathrm{d}\theta, \tag{A11}$$

where (3.35), (3.36) have been used. The potential $\Upsilon^{(2)}$ on the cylinder surface is determined in (3.37). We can easily evaluate the integral (A11)

$$\bar{F}_{s1} = -\pi \sum_{n=1}^{\infty} n(n+1)(Z_n Z_{n+1}^* + Z_{n+1} Z_n^*),$$

$$\bar{F}_{s2} = -\frac{\pi}{\gamma} \sum_{n=1}^{\infty} n[2n|Z_n|^2 + \mathrm{i}d(n+1)(Z_n Z_{n+1}^* - Z_{n+1} Z_n^*)].$$

These series converge as R^{2n} $(R < 1)$.

A.1. *The cylinder in the upper layer*

In dimensionless variables, using the conformal mapping (4.1), we may write the potentials $\Upsilon^{(s)}$ as the series

$$\Upsilon^{(1)} = \mathrm{Re}\left\{\sum_{n=1}^{\infty} Z_n(\rho^n + \rho^{-n})\, \mathrm{e}^{in\theta}\right\}, \qquad \Upsilon^{(2)} = \mathrm{Re}\left\{\sum_{n=1}^{\infty} Z_n R_n^- R^{-n} \rho^n \mathrm{e}^{in\theta}\right\}. \tag{A12}$$

The recursive relations for the coefficients in (A12) are as follows:

$$\begin{aligned} &Z_{n-2}(n-2)(n-1)\tilde{P}_{n-2} - 2Z_{n-1} i(n-1)[(2n-1)\tilde{P}_{n-1} - \varepsilon\mu\gamma R_{n-1}^-] \\ &\quad -2Z_n n(3n\tilde{P}_n - 2\varepsilon\mu\gamma R_n^-) + 2Z_{n+1} i(n+1)[(2n+1)\tilde{P}_{n+1} - \varepsilon\mu\gamma R_{n+1}^-] \\ &\quad +Z_{n+2}(n+2)(n+1)\tilde{P}_{n+2} = 0. \end{aligned} \tag{A13}$$

All notations are identical to those of Section 4. To determine the coefficients Z_1 and Z_2 we introduce new functions (A3), and for their sum with weights

$$\bar{W}_2(\xi) = \varepsilon\bar{F}_1(\xi) - (2+\varepsilon)\bar{F}_2(\xi) \tag{A14}$$

we obtain a differential equation

$$(1+i\xi)^4 \bar{W}_2'' + 2i(1+i\xi)^2(1+i\xi - \bar{\mu}\gamma)\bar{W}_2' = \bar{K}_0 + 4i\bar{\mu}\gamma(1+\varepsilon)(1+i\xi)^2 \bar{F}_1',$$

where

$$\tilde{K}_0 = 2\{Z_2\tilde{P}_2 + iZ_1[\tilde{P}_1 + 2\gamma^2\bar{\mu}(2+\varepsilon)]\}.$$

The solution for $\bar{W}_2'$ has the form

$$\bar{W}_2' = \frac{1}{(1+i\xi)^2}\left\{\frac{i\tilde{K}_0}{2\beta}\left(1-\exp\frac{2i\beta\xi}{1+i\xi}\right)\right.$$
$$\left. +2i\beta\left[2\bar{F}_1+\sum_{n=1}^{\infty} Z_n R^n J_n(\beta,\xi)\right]+\tilde{c}_0\exp\frac{-2\beta}{1+i\xi}\right\},$$

where $\tilde{c}_0$ is the constant of integration. From (A14) $\bar{W}_2'(0) = Z_1\tilde{P}_1$ and we have $\tilde{c}_0 = Z_1\tilde{P}_1\,\mathrm{e}^{2\beta}$.

Let us consider a behaviour of the x-derivative of $\Upsilon^{(1)}$ at the interface

$$\left.\frac{\partial\Upsilon^{(1)}}{\partial x}\right|_{y=-d} = \left.\frac{\partial\Upsilon^{(1)}}{\partial\theta}\frac{\partial\theta}{\partial x}\right|_{\rho=R} = -\frac{(1+i\,\mathrm{e}^{i\theta})}{2\gamma}\sum_{n=1}^{\infty} nZ_nR_n^+\,\mathrm{e}^{i(n-1)\theta}$$
$$= \frac{(1+i\,\mathrm{e}^{i\theta})^2}{2\gamma(2+\varepsilon)}[\bar{W}_2'-2(1+\varepsilon)\bar{F}_1']. \tag{A15}$$

The latter equality in (A15) follows from the form of $\bar{W}_2'$ at $\xi = \mathrm{e}^{i\theta}$ according to (A14). According to the radiation condition $\partial\Upsilon^{(1)}/\partial x \to -1$ at $x \to \infty$ and we have the linear second-order system for the definition of Z_1 and Z_2

$$Z_1\left[2i\beta\sum_{n=1}^{\infty} A_nR^nS_n^+(\beta)+\tilde{P}_1\right]+Z_2 2i\beta\sum_{n=1}^{\infty} B_nR^nS_n^+(\beta) = -2\gamma(2+\varepsilon), \tag{A16}$$

$$Z_1[\tilde{P}_1+2\gamma\beta(2+\varepsilon)]-Z_2 iP_2 = 2\gamma\beta(2+\varepsilon). \tag{A17}$$

Here A_n and B_n are computed from the recursive relations (A13) at $Z_1 = 1,\ Z_2 = 0$ and $Z_1 = 0,\ Z_2 = 1$, respectively. On solving (A16), (A17) with (A10), we can define all coefficients of the series (A12).

Computation of the steady loads (2.10) is performed in the following way

$$\tilde{F}_{sj} = \frac{F_{sj}}{\rho_1 aU^2} = -\frac{1}{2}\int_0^{2\pi}\left(\frac{\partial\Upsilon^{(1)}}{\partial\theta}\right)^2\lambda n_j\,\mathrm{d}\theta.$$

The potential $\Upsilon^{(1)}$ on the cylinder surface is determined in (4.13) and the resultant expressions are

$$\tilde{F}_{s1} = \pi\sum_{n=1}^{\infty} n(n+1)(Z_nZ_{n+1}^*+Z_{n+1}Z_n^*),$$

$$\tilde{F}_{s2} = \frac{\pi}{\gamma}\sum_{n=1}^{\infty} n[2n|Z_n|^2+id(n+1)(Z_nZ_{n+1}^*-Z_{n+1}Z_n^*)].$$

In Table 4 we give the wave resistance and lift on a circular cylinder with $h = 2a$ and $\varepsilon = 0{\cdot}03$. The Froude number is here defined as $\overline{\mathrm{Fn}} = U/\sqrt{gh}$. For a homogeneous fluid with

Table 4. Wave resistance and lift on a circular cylinder with $h = 2a$.

	Homogeneous fluid with free surface		Two-layer fluid with $\varepsilon = 0{\cdot}03$			
			cylinder in lower layer		cylinder in upper layer	
$\overline{\mathrm{Fn}}$	$-\bar{F}_{s1}$	$\bar{F}_{s2}$	$-\bar{F}_{s1}$	$\bar{F}_{s2}$	$-\tilde{F}_{s1}$	$\tilde{F}_{s2}$
0·0	0·00000	0·22978	0·00000	0·22978	0·00000	−0·22978
0·2	0·00000	0·26426	0·00000	0·24713	0·00000	−0·24662
0·4	0·00724	0·45068	0·00332	0·33640	0·00322	−0·33314
0·6	0·65972	0·87870	0·28947	0·52419	0·27981	−0·51465
0·8	1·03160	0·14388	0·51431	0·18746	0·49886	−0·18874
1·2	0·34919	−0·37258	0·20619	−0·13184	0·20110	0·12309
1·6	0·10634	−0·31638	0·06431	−0·10596	0·06278	0·09815
2·0	0·03727	−0·26041	0·02239	−0·06969	0·02186	0·06267
∞	0·00000	−0·16956	0·00000	−0·00290	0·00000	−0·00291

a free surface we have $\bar{g} \to g$ and $\overline{\mathrm{Fn}} \to \mathrm{Fn} = U/\sqrt{gh}$. The numerical results for steady hydrodynamic loads in a homogeneous fluid are in a complete agreement with the results of [4], where only the range $0{\cdot}5 \leqslant \mathrm{Fn} \leqslant 1$ was considered. A graphic presentation of wave resistance and lift on a circular cylinder located in the upper or lower layer of a two-layer fluid was presented by Wu [12]. Our solution has confirmed these results.

The wave resistance as a function of $\overline{\mathrm{Fn}}$ is shown for all three cases in Table 4. They are seen to be similar. The behaviour of the lift depends on location of the cylinder above or below the interface. Near a solid plane ($\overline{\mathrm{Fn}} \to 0$) the cylinder is attracted to the plane. In another limiting case of weightless fluid the vertical forces are always directed downwards, *i.e.* the sinking force acts on the body regardless of its location.

Appendix B. The special integrals

Through partial integration, the integrals J_n in (3.22) are shown to be connected recurrently starting from the two integrals J_0, J_1. We have

$$J_{n+1} = 2i J_n - \frac{2i\beta}{n}(2\xi^n + J_n) + J_{n-1} \quad (n \geqslant 1), \tag{B1}$$

$$J_0 = 2\left[\exp\left(\frac{2i\beta\xi}{1+i\xi}\right) - 1\right], \tag{B2}$$

$$J_1 = 4i\beta \exp\left(-\frac{2\beta}{1+i\xi}\right)\left[i\pi \operatorname{sign}\left(\operatorname{Im}\frac{2\beta}{1+i\xi}\right) - E_1\left(-\frac{2\beta}{1+i\xi}\right) - \mathrm{Ei}(2\beta)\right] + i J_0. \tag{B3}$$

We have used the standard definitions of the exponential integrals Ei and E_1 as given by Abramowitz and Stegun [13]). The function E_1 has a branch cut along the negative real axis. Assuming $\xi = R/w^*$, we obtain:

for a cylinder located in the lower layer, according to (3.1)

$$\frac{1}{1+i\xi} = \frac{1}{2\gamma R}(1 - iRx - Ry), \qquad \frac{i\xi}{1+i\xi} = \frac{1}{2\gamma}(ix + y - R); \tag{B4}$$

for a cylinder located in the upper layer, according to (4.1)

$$\frac{1}{1+i\xi} = \frac{1}{2\gamma R}(1 + iRx + Ry), \qquad \frac{i\xi}{1+i\xi} = -\frac{1}{2\gamma}(ix + y + R). \tag{B5}$$

Defining the new functions $S_n(\beta, \xi)$ by formulae

$$J_n(\beta, \xi) = \exp\left(\frac{2i\xi\beta}{1+i\xi}\right) S_n(\beta, \xi) - 2i^n,$$

we have from (B1)–(B5) at $|x| \to \infty$ ($\xi \to i$, $w \to R$)

$$S_{n+1} = S_{n-1} + 2i(1 - \beta/n)S_n, \tag{B6}$$

$$S_0 = 2, \qquad S_1 = 2i + 4\beta\, \mathrm{e}^{-2\beta}[\delta\pi\ \mathrm{sign}(x) - i\ \mathrm{Ei}(2\beta)], \tag{B7}$$

where $\delta = +1$ and $\delta = -1$ for the cylinder located in the lower and upper layers, respectively. From (B6), (B7) it is evident that at $|x| \to \infty$ for the given location of the cylinder, the S_n depend only on β and $\mathrm{sign}(x)$. Here we use the notation $S_n^{\pm}(\beta)$, where the upper signs '$+$' and '$-$' correspond to $\mathrm{sign}(x)$.

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